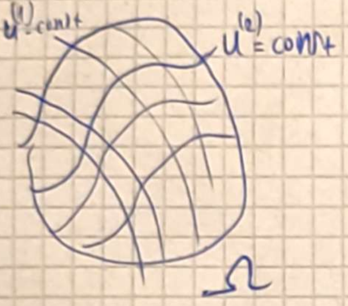
Differential Geometry I | Week 7

A quick recap on curvilinear coordinates

If $\Omega \subseteq \mathbb{R}^n$ is open: The functions $u_1, \dots, u_n : \Omega \to \mathbb{R}$ define a (curvilinear) system of coordinates on Ω if the Jacobian matrix



$$\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots \\ \frac{\partial u_2}{\partial x_1} & \cdot & \\ \vdots & & \cdot \end{pmatrix}$$ is invertible everywhere

Equivalently: If $\Phi : \Omega \to \mathbb{R}^n$ is defined by

$\Phi(x_1, \dots, x_n) = (u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n))$, then Φ is a diffeomorphism from Ω to $\Phi(\Omega)$.

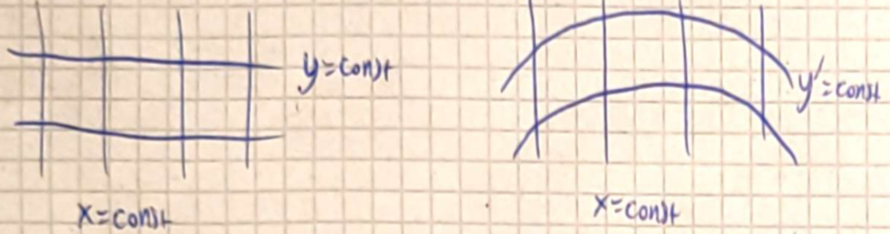
We call the transition $(x_1, \dots, x_n) \to (u_1, \dots, u_n)$: Change of coordinates.

Fundamental example: On $\mathbb{R}^2 / ((-\infty, 0] \times \{0\})$: Polar coordinates r, θ.

- Sometimes: We choose a coordinate system conveniently adapted to a particular setting, e.g.:
  If $\Omega = \{1 < \sqrt{x^2 + y^2} < 2\}$, then switching to polar coordinates $\Omega = \{1 < r < 2\}$ has a simpler expression

Caution: When considering a partial derivative like $\frac{\partial f}{\partial x}$, one has to specify the whole coordinate system, since $\frac{\partial f}{\partial x}$ means "differentiating in x while keeping the rest of the coordinates fixed"

For instance: On $\mathbb{R}^2$: In the coordinate systems (x,y) and $(x,y') = (x, y+x^2)$:



If $f(p) = (\mathrm{dist}(0,p))^2$ then

- In (x,y): $f(x,y) = x^2+y^2$
- In (x,y'): $f(x,y') = x^2 + (y'-x^2)^2$

At the point $q = \begin{cases} (1,1) & \text{in } (x,y) \\ (1,2) & \text{in } (x,y') \end{cases}$ (same point, different coordinates in the two coordinate systems)

$$\left.\frac{\partial f}{\partial x}\right|_q = \begin{cases} 2 & \text{in } (x,y) \text{ coordinates} \\ -2, & \text{in } (x,y') \text{ coordinates.} \end{cases}$$

Recall the <u>inverse function theorem</u>: (From Analysis II)

<u>Theorem</u>: Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^k(\Omega, \mathbb{R}^n)$ with $k \geq 1$. If, for some $p \in \Omega$, $J_f(p) \neq 0$, then f is a local diffeom. (of class C^k) around p.

Also recall: A linear map $\ell : \mathbb{R}^m \to \mathbb{R}^n$ has rank r if there exists bases in $\mathbb{R}^m$ and $\mathbb{R}^n$ such that the matrix expression of ℓ takes the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

or, in the corresponding coordinates:

$$\ell(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0)$$

<u>Definition</u>: Let $U \subseteq \mathbb{R}^m$ be open and $f \in C^1(U, \mathbb{R}^n)$.

Rank of f: $\text{Rank}_f : U \to \mathbb{N}$ is $\text{Rank}(df_p)$

- f is of <u>maximal rank</u> at p if $\text{Rank}_f(p) = \min(m,n)$
- f is an <u>immersion</u> if, $\forall p \in U$, df_p is <u>injective</u>
  ($\Leftrightarrow \text{Rank}_f(p) = m$)
- f is a <u>submersion</u> if, $\forall p \in U$, df_p is <u>surjective</u>
  ($\Leftrightarrow \text{Rank}_f(p) = n$)

<u>Example</u>:

- If $f: \mathbb{R} \to \mathbb{R}^2$ is $f(x) = (x^2, x^3)$ then

$$Df = \begin{bmatrix} 2x \\ 3x^2 \end{bmatrix} \quad \text{so} \quad \text{Rank}_f = \begin{cases} 0 & \text{at } x = 0 \\ 1 & \text{at } x \neq 0 \end{cases}$$

(A regular curve is always an immersion)

- If $G: \mathbb{R}^2 \to \mathbb{R}^2$ given by $G(x,y) = (x^2, y)$ ("Folding" map)

then $$DG = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix} \quad \text{so} \quad \text{Rank}_G(x,y) = \begin{cases} 1, & x = 0 \\ 2, & x \neq 0 \end{cases}$$

<u>Lemma</u>:

If $U \subseteq \mathbb{R}^m$ is open and $f \in C^1(U, \mathbb{R}^n)$, then Rank_f is <u>lower-semicontinuous</u> on U, i.e. if $\text{Rank}_f(p) = r$, then $\text{Rank}_f \geq r$ in a neighborhood of p.

<u>Proof</u>: Since $\text{Rank}_f(p) = r$: The Jacobian matrix $Df(p)$

has rank $r \Longrightarrow \exists$ $r \times r$ minor of $DF(p)$ with determinant $\neq 0$. Since DF is continuous, the same minor has non-zero determinant close enough to p, so the rank of DF is $\geq r$ near p. $\square$

<u>Constant rank theorem:</u> Let $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a C^k map of <u>constant rank</u> $= r$. Then, $\forall p \in U$, there exists a neighborhood V of p and a neighborhood W of $q = f(p)$, as well as C^k diffeomorphisms

$$\phi: V \to V' \subseteq \mathbb{R}^m,$$
$$\psi: W \to W' \subseteq \mathbb{R}^n$$

such that:

i) $f(V) \subseteq W$

ii) $\phi(p) = 0_{\mathbb{R}^m}$ and $\psi(q) = 0_{\mathbb{R}^n}$

iii) The map $F = \psi \circ f \circ \phi^{-1}: V' \to W'$ takes the form

$$F(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0).$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi"] & W \arrow[d, "\psi"] \\ V' \arrow[r, "F"] & W' \end{tikzcd}$$

<u>Remark:</u> ϕ, ψ define curvilinear coordinates. The above is the statement that, in these coordinates, f takes the simple form above.

"Flat"

Proof: Assume w.l.o.g that $p = 0$ and $q = f(p) = 0$.

After performing a changes of basis in $\mathbb{R}^m$ and $\mathbb{R}^n$:

Assume that $DF(0) = \left[\frac{\partial f_i}{\partial x_j}(0)\right] = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

(possible since $\operatorname{rank} DF(0) = r$)

Define the map $\phi : V \to V'$ by

$$\phi(x_1, \dots, x_m) = (f_1(x), \dots, f_r(x), x_{r+1}, \dots, x_m)$$

Then

$$D\phi(0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} & \cdots & \frac{\partial f_r}{\partial x_m} \\ 0 & \cdots & 0 & 1 \; 0 \cdots & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & & 1 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{m-r} \end{bmatrix} = I$$

So by the implicit function theorem: $\exists$ neighborhood V of 0 in U such that $\phi : V \to \phi(V) = V'$ is a diffeomorphism (of class C^k).

And: $f \circ \phi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_r, f_{r+1} \circ \phi^{-1}(x), \dots, f_m \circ \phi^{-1}(x))$.

The Jacobian matrix of $f \circ \phi^{-1}$ is a $m \times n$ matrix of the form:

$$D(f \circ \phi^{-1})(x) = \begin{pmatrix} I_r & 0 \\ * & \Delta(x) \end{pmatrix} \quad \text{since} \quad d(f \circ \phi^{-1}) = df \circ d\phi^{-1}$$

and $\operatorname{rank}(df) = \text{const} = r$, $\operatorname{rank}(d\phi^{-1}) = \max$: $f \circ \phi^{-1}$ also has $\operatorname{rank} = r$, so $\Delta(x)$ has to be the zero matrix

Bw: $\Delta(x) = \left(\frac{\partial}{\partial x_j} (f_i \circ \phi^{-1})(x) \right)_{\substack{r+1 \leq j \leq m \\ r+1 \leq i \leq n}}$

so, for $i \geq r+1$: $f_i \circ \phi^{-1}$ is <u>independent</u> of $x_{r+1}, \dots, x_m$

So we can write

$$f \circ \phi^{-1}(x) = (x_1, \dots, x_r, h_{r+1}(x_1, \dots, x_r), \dots, h_n(x_1, \dots, x_r))$$

Define the map $\Psi: f(U) \to \mathbb{R}^n$ by

$$\Psi(y_1, \dots, y_n) = (y_1, \dots, y_r, y_{r+1} - h_{r+1}(y_1, \dots, y_r), \dots, y_n - h_n(y_1, \dots, y_r))$$

Note that, at 0:

$$D\Psi(0) = \begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}$$ so there exists a neighborhood W of 0 such that $\Psi: W \to \Psi(W) = W'$ is a diffeomorphism of class C^k.

Then $F = \Psi \circ f \circ \phi^{-1}$ has the required form.

By shrinking V if necessary: We also have $f(V) \subseteq W$. $\square$

<u>Corollary: Implicit function theorem</u>

Let $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be of class C^k and $m = n + \ell$, $\ell \geq 1$.

Assume that, at $p \in U$, the minor of the Jacobian

$$\left(\frac{\partial f_i}{\partial x_j}(p) \right)_{\substack{1 \leq i \leq n \\ \ell+1 \leq j \leq m}}$$ is invertible.

Then there exists a neighborhood of p of the form $V \times W \subseteq U$ with $V \subseteq \mathbb{R}^l$, $W \subseteq \mathbb{R}^n$ and a map $\phi: V \to W$ of class C^k such that, for all $x \in V \times W$ with $f(x) = f(p)$, we have

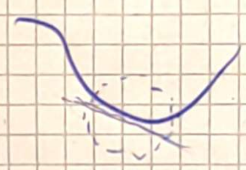
$$(x_{l+1}, \dots, x_m) = \phi(x_1, \dots, x_l).$$

Submanifolds of $\mathbb{R}^n$

Informally, a submanifold of $\mathbb{R}^n$ of dimension m locally "looks" like an affine subspace (hyperplane) of dimension m.

Prototypical examples to keep in mind:

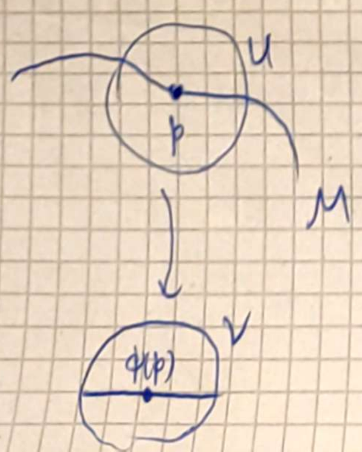
- A regular curve in $\mathbb{R}^n$:



- A surface in $\mathbb{R}^3$:

Definition:

A subset $M \subseteq \mathbb{R}^n$ is a submanifold of dimension $m \leq n$ and of class C^k if $\forall p \in M$, $\exists$ open neighborhood $U \subseteq \mathbb{R}^n$ of p and a C^k diffeomorphism $\phi: U \to V \subseteq \mathbb{R}^n$ such that $\phi(U \cap M) = V \cap E$, where E is a vector space of dimension m.



~~Remark~~:

i) <u>Note:</u> $n-m$ is the codimension of M

- dimension 1: Curve
- dimension 2: surface
- Codimension 1: Hypersurface.

<u>Remark:</u> By composing ϕ with an affine map, we can always assume w.l.o.g that $\phi(p)=0$ and E is the subspace defined (in the Cartesian coordinates of $\mathbb{R}^n$) by $\{y_{m+1}=y_{m+2}=\cdots=y_n=0\}$.

The above definition is equivalent to the statement that there exists a set of curvilinear coordinates $\{y_1,\dots,y_n\}$ around p such that

$$U\cap M \text{ is the set } \{y_{m+1}=\dots=y_n=0\}.$$

Then $y_k=\phi_k(x_1,\dots,x_n)$ is the change of coordinates.

So locally: M is the solution of $n-m$ (non-linear) equations in the original coordinates x.

<u>Examples:</u>

i) A submanifold of dimension 0: A discrete set of points in $\mathbb{R}^n$

ii) A submanifold of dimension n: An open subset of $\mathbb{R}^n$

iii) If $f: U\to\mathbb{R}$ is C^k: The graph

$$M=\{(x,t)\in U\times\mathbb{R}: t=f(x)\}$$

is a C^k submanifold of $\mathbb{R}^{n+1}$, of codimension 1.

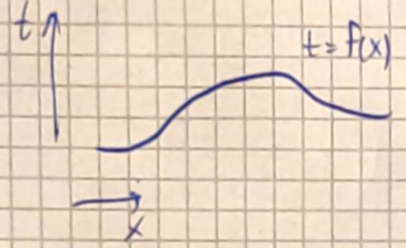
Since the map $\phi : U \times \mathbb{R} \to U \times \mathbb{R}$,

$$\phi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, x_{n+1} - f(x_1, \dots, x_n))$$

is a diffeomorphism of class C^k,

(with inverse $\phi^{-1}(y_1, \dots, y_n, y_{n+1}) = (y_1, \dots, y_n, y_{n+1} + f(y_1, \dots, y_n))$)

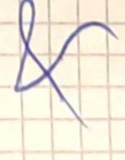
and $\phi((U \times \mathbb{R}) \cap M) = U \times \{0\} = (U \times \mathbb{R}) \cap (\{y_{n+1} = 0\})$



<u>Theorem</u>: If $U \subseteq \mathbb{R}^m$ open and $f \in C^k(U, \mathbb{R}^n)$ of constant rank r, then:

A) $\forall q \in \mathbb{R}^n$, the preimage $f^{-1}(q) \subseteq U$ is a submanifold of <u>codimension r</u>

B) $\forall p \in U$, $\exists$ neighborhood $V_p \subseteq U$ such that the image $f(V_p)$ is a submanifold of $\mathbb{R}^n$ of <u>dimension r</u>.

(Note: If $f : U \to \mathbb{R}^n$ is an immersion: $f(U)$ is not alway a submanifold, e.g. a self-intersecting curve )

Remark: This will be our main tool of "constructing" submanifolds.

Special case: If $f: U \subseteq \mathbb{R}^m \to \mathbb{R}$ satisfies $Df \neq 0$ (so Df: rank 1), then $\{f = \text{const}\}$ is a codimension 1 submanifold of $\mathbb{R}^m$.

Example(s): • The unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is a smooth hypersurface (it's the level set of $\sum_{i=1}^{n} x_i^2 = 1$).

• If $F, G: \mathbb{R}^n \to \mathbb{R}$ are such that DF, DG are linearly independent: The set $\{F = \text{const}\} \cap \{G = \text{const}\}$ is a codimension 2 submanifold.

• The set $O(n) \subset M_n(\mathbb{R})$ is a submanifold (defined by the equation $X^T X = \mathbb{I}$).

Proof of the theorem:

By the constant rank theorem: For every $p \in U$, $\exists$ neighborhood V of p and W of $q = f(p)$, as well as C^k diffeomorphisms $\phi: V \to V' \subseteq \mathbb{R}^m$, $\psi: W \to W' \subseteq \mathbb{R}^n$ such that

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0).$$

and $\phi(p) = 0$, $\psi(f(p)) = 0$.

Case A): The set $M = \{x \in U : f(x) = f(p)\} \subseteq \mathbb{R}^m$

satisfies: $M \cap V = \{x \in V : \psi \circ f(x) = 0\}$

So $\Phi(M \cap V) = \{\Phi(x) : x \in V, \ \psi \circ f(x) = 0\}$

$= \{y \in V' : \psi \circ f \circ \Phi^{-1}(y) = 0\}$

$= \{y \in V' : y_1 = \dots = y_r = 0\}$

(So Φ is the ~~[illegible]~~ diffeomorphism "flattening out" M)

Case B) The set $f(V) \subseteq \mathbb{R}^n$

$M' =$

satisfies: $\psi(M') = \{\psi(y) : y = f(x) \text{ for some } x \in V\}$

$= \{\psi \circ f(x) : x \in V\}$

$= \{\psi \circ f \circ \Phi^{-1}(z) : z \in V'\}$

$= \{(z_1, \dots, z_r, 0, \dots, 0) : z \in V'\}$

So $\psi(M')$ is the r-dimensional subspace defined by $y_{r+1} = \dots = y_n = 0$.